Ejercicios temas 3 y 4

1. Hallar la solución general y la matriz fundamental de los siguientes sistemas:

   (a) $x' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} x$

   Sol: $x(t) = (C_2t + C_1)e^t, y(t) = C_3e^{2t}, z(t) = C_2e^t$

   (b) $x' = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} x$

   Sol: $x(t) = (2C_2t + C_1)e^{2t}, y(t) = C_3e^{2t}, z(t) = C_2e^{2t}$

   (c) $x' = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & -1 \\ 0 & 0 & 3 \end{pmatrix} x$, con $x(0) = (2, 0, -1)$.

   Sol: $x(t) = 2e^{3t}, y(t) = 3te^{3t}, z(t) = -e^{3t}$

   (d) $x' = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 2 & 2 \end{pmatrix} x$

   Sol: $x(t) = C_3e^{2t}, y(t) = (C_3t + C_2)e^{2t}, z(t) = C_4e^{2t}, v(t) = (-C_3t + 2C_4t + C_1)e^{2t}$.

   (e) $x' = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} x$

   Sol: $x(t) = (3C_2t + C_1)e^t, y(t) = C_2e^t$.

   (f) $x' = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} x$

   Sol: $x(t) = e^{2t}(C_2cos(t) + C_1sen(t)), y(t) = e^{2t}(C_1cos(t) - C_2sen(t))$.

   (g) $x' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} x$

   Sol: $x(t) = (C_3t + C_2)e^{2t}, y(t) = C_3e^{2t}, z(t) = (-C_3t + C_1)e^{2t}$.

   (h) $x' = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 1 & 2 \end{pmatrix} x$

   Sol: $x(t) = C_3e^{2t}, y(t) = (C_3t + C_2)e^{2t}, z(t) = (\frac{1}{2}(C_3t^2 + 2C_2t - 2C_3t + 2C_1))e^{2t}$.

(i) $x' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x$
Sol: $x(t) = C_1 e^{3t} + C_2 e^{t}, y(t) = C_1 e^{3t} - C_2 e^{t}.$

(j) $x' = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} x$
Sol: $x(t) = C_3 e^{2t} + C_2 e^{t} t + C_1 e^{t}, y(t) = C_4 e^{2t}, z(t) = C_3 e^{2t} + C_2 e^{t}, v(t) = C_3 e^{2t}.$

(k) $x' = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} x$
Sol: $x(t) = C_2 e^{3t} t + C_1 e^{3t} + \frac{1}{16} C_3 e^{-t}, y(t) = C_4 e^{-t}, z(t) = -\frac{1}{4} C_3 e^{-t} + C_2 e^{3t}, v(t) = C_3 e^{-t}.$

(l) $x' = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} x$
Sol: $x(t) = (2C_3 t + C_2) e^{t}, y(t) = C_4 e^{t}, z(t) = (-C_3 t + C_1) e^{t}, v(t) = C_3 e^{t}.$

(m) $x' = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} x$
Sol: $x(t) = -C_3 e^{2t} + C_2 e^{3t}, y(t) = (C_4 t + C_1) e^{3t} - C_3 e^{2t}, z(t) = C_4 e^{3t}, v(t) = C_3 e^{2t}.$

(n) $x' = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} x$
Sol: $, x(t) = (C_2 t + C_1) e^{2t}, y(t) = C_4 e^{2t}, z(t) = C_3 e^{2t}, v(t) = C_2 e^{2t}.$

(o) $x' = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} x$
Sol:$x(t) = C_3 e^{3t}, y(t) = (C_2 t + C_1) e^{3t}, z(t) = C_2 e^{3t}, v(t) = C_4 e^{3t}$

(p) $x' = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x$, con $x(0) = (1, 0, 3, 1)$.
Sol:$x(t) = -e^{t} + 2e^{2t}, y(t) = 3te^{2t}, z(t) = 3e^{2t}, v(t) = e^{t}.$

(q) $x' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 3 \end{pmatrix} x$
Sol: $x(t) = C_3 e^{3t}, y(t) = (\frac{1}{2}(C_3 t^2 + 2C_2 t + 2C_1)) e^{3t}, z(t) = (C_3 t + C_2) e^{3t}.$

(r) $x' = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x$

Sol: $x(t) = (2C_3t + C_2)e^t, y(t) = (C_3t^2 + C_2t + C_1)e^t, z(t) = C_3e^t$.

2. Sea
$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$
hallar e^{At}.
Sol:$e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2}t^2e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{pmatrix}$

3. Resolver los sistemas, clasificar el punto fijo $(0, 0)$, determinar su estabilidad y dibujar el plano de fases:

   (a) $x' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} x$
   Sol: $x(t) = C_1e^{-t} + C_2e^{2t}, y(t) = 2C_1e^{-t} + \frac{1}{2}C_2e^{2t}$.

   (b) $x' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x$
   Sol: $x(t) = e^{-t}(C_2cos(t) + C_1sen(t)), y(t) = -\frac{1}{5}e^{-t}(C_1cos(t) - 2C_2cos(t) - 2C_1sen(t) - C_2sen(t))$

   (c) $x' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$
   Sol: $x(t) = C_2e^t, y(t) = C_1e^t$.

   (d) $x' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x$
   Sol: $x(t) = (C_2t + C_1)e^t, y(t) = \frac{1}{4}e^t(2C_2t + 2C_1 - C_2)$.

   (e) $x' = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} x$
   Sol: $x(t) = C_1e^{2t} + C_2e^t, y(t) = \frac{3}{2}C_1e^{2t} + C_2e^t$.

   (f) $x' = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} x$
   Sol: $x(t) = C_2e^{-t} + C_1e^{-2t}, y(t) = C_2e^{-t}$.

   (g) $x' = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} x$
   Sol:$x(t) = \frac{1}{3}C_2e^{2t} + C_1e^{-t}, y(t) = C_2e^{2t})$

   (h) $x' = \begin{pmatrix} 1 & -6 \\ 1 & -4 \end{pmatrix} x$
   Sol:$x(t) = C_1e^{-t} + C_2e^{-2t}, y(t) = \frac{1}{3}C_1e^{-t} + \frac{1}{2}C_2e^{-2t}$.

   (i) $x' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x$
   Sol: $x(t) = C_1e^{3*t} + C_2e^t, y(t) = C_1e^{3t} - C_2e^t$.

   (j) $x' = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} x$
   Sol:$x(t) = -C_2e^{-t} + C_1e^t, y(t) = C_2e^{-t}$.

(k) $x' = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} x$
Sol:$x(t) = C_2e^{2t} + C_1e^t, y(t) = C_2e^{2t}$.

(l) $x' = \begin{pmatrix} 0 & 3 \\ -1 & -4 \end{pmatrix} x$
Sol:$x(t) = C_1e^{-3t} + C_2e^{-t}, y(t) = -C_1e^{-3t} - \frac{1}{3}C_2e^{-t}$.

4. Resolver el sistema usando el método de Euler

$$\left.\begin{array}{r} x' = ty, \\ y' = x, \\ x(0) = 1, y(0) = 0, \end{array}\right\}$$

en el intervalo $[0, 1]$. Usar $h = 0.5$. ¿De qué orden es el método? ¿Qué error esperamos si cogemos $h = 0.25$?

5. Resolver el sistema usando el método de Euler

$$\left.\begin{array}{r} x' = 2y, \\ y' = t + x, \\ x(1) = 1, y(1) = 0, \end{array}\right\}$$

en el intervalo $[1, 2]$. Usar $h = 0.5$. ¿De qué orden es el método? ¿Qué error esperamos si cogemos $h = 0.25$?

6. Resolver el problema

$$\left.\begin{array}{r} y''' - y' = e^{3t}, \\ y(0) = 0, y'(0) = 1, y''(0) = 0, \end{array}\right\}$$

usando la transformada de Laplace.
Sol:$y(t) = -\frac{5}{8}e^{-t} + \frac{1}{4}e^t + \frac{1}{24}e^{3*t} + \frac{1}{3}$.

7. Hallar la transformada de la función $f(t) = t^2sen(3t)$ y la trasnformada inversa de $F(s) = \frac{1}{s^2+2s+4}$.
Sol: $\mathcal{L}(f(t)) = \frac{(18(s^2-3))}{(s^2+9)^3}$. $\mathcal{L}^{-1}(F(s)) = \frac{\sqrt{3}}{3}e^{-t}sen(\sqrt{3}t)$.

8. Resolver el problema

$$\left.\begin{array}{r} y''' - y'' = e^{4t}, \\ y(0) = 1, y'(0) = 0, y''(0) = 0, \end{array}\right\}$$

usando la transformada de Laplace.
Sol: $y(t) = \frac{1}{48}e^{4t} - \frac{1}{3}e^t + \frac{1}{4}t + \frac{5}{16}$.

9. Hallar la transformada de la función $f(t) = t^3e^{3t}$ y la trasnformada inversa de $F(s) = \frac{1}{s^2-2s+5}$.
Sol: $\mathcal{L}(f(t)) = \frac{6}{(s-3)^4}$, $\mathcal{L}^{-1}(F(s)) = \frac{1}{2}e^tsen(2t)$.

10. Resolver el problema

$$\left.\begin{array}{r} y''' + y'' + y' + y = 0, \\ y(0) = 0, y'(0) = 0, y''(0) = 1, \end{array}\right\}$$

usando la transformada de Laplace.
Sol: $y(t) = \frac{1}{2}e^{-t} + \frac{1}{2}sen(t) - \frac{1}{2}cos(t)$.

11. Resolver el problema

$$\left.\begin{array}{r} y''' - y = 0, \\ y(0) = 2, y'(0) = 0, y''(0) = 1, \end{array}\right\}$$

usando la transformada de Laplace.
Sol: $y(t) = e^t - \frac{1}{3}\sqrt{3}e^{-\frac{t}{2}}sen(\frac{\sqrt{3}t}{2}) + e^{-\frac{t}{1}}cos(\frac{\sqrt{3}t}{2})$.

12. Sabiendo que $\mathcal{L}(y(t)) = \frac{1}{s^5}$, usando las propiedades de la transformada de Laplace, hallar $\mathcal{L}(t^3 y(t))$ y $\mathcal{L}(e^{2t}y(t))$.

13. Sabiendo que $\mathcal{L}(y(t)) = \frac{1}{1+s^2}, y(0) = 4, y'(0) = 1$, usando las propiedades de la transformada de Laplace, hallar $\mathcal{L}(e^{-3t}y(t)), \mathcal{L}(ty''(t))$.

14. Hallar la transformada inversa de Laplace de $F(s) = \frac{s}{s^2+4s+8}$.
Sol:$\mathcal{L}^{-1}(F(s)) = e^{-2t}(cos(2t) - sen(2t))$

15. Sabiendo que $\mathcal{L}(y) = \frac{1}{s^2+1}, y(0) = 1, y'(0) = 2$, usando las propiedades de la transformada de Laplace, hallar $\mathcal{L}(ty'''), \mathcal{L}(e^{-2t}y(t))$.